Math 155, ***Lecture Notes- Bonds*** **Name____________**

Section 9.9 *Representation of Functions by Power Series*

In this section, we will consider a few interesting techniques that will allow us to find a power series that represents a given function. In particular, we will focus on using the formula for the sum of a convergent geometric series to define a power series representation of a particular function. If needed we can move the center of the series, we can perform algebraic operations with a series, or combinations of series, or we can use calculus based operations like differentiation, or integration to create a particular series representation of a given function.

From Section 9.2, we can recall the following theorem:

THEOREM 9.6 Convergence of a Geometric Series

A geometric series with ratio r diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

Ex. 1: If we let $a=1$ and $r=x$, the geometric series sum formula gives us a power series representation for $f(x)=\dfrac{1}{1-x}$ centered at $c=0$.

That is, $f(x)=\dfrac{1}{1-x}=\sum_{n=0}^{\infty}(1)(x)^n=\sum_{n=0}^{\infty}x^n$, for $|x|<1$. This series converges absolutely on $(-1,1)$.

We will use this geometric power series sum formula to develop many other representations of functions by manipulating values of a, r, and c.

Ex. 2: Use the geometric series sum formula to represent $f(x)=\frac{1}{1-x}$ as a power series centered at $c=-1$, and find the domain of this power series function.

When we change the center of this power series, we should see $(x+1)^{?}$, which will show the new center at $c=-1$. Also, we will be able to find a corresponding change in the domain of the power series representation, since we will be moving the center of the previous interval of convergence, $(-1,1)$.

More Ex. 2:

Ex. 3: Use the geometric series sum formula to represent $f(x)=\dfrac{4}{3x+2}$ as a power series centered at $c=2$, and find the interval of convergence (domain) of this power series function.

More Ex. 3:

Operations with Power Series

Let $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$.

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$

2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$

3. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

NOTE:
- For simplicity, the properties are stated for series centered at $c = 0$.
- These operations can change the interval of convergence.
- When two series are summed, the interval of convergence for the sum is the intersection of the intervals of convergence of two original series.

Ex. 4: Use the geometric series sum formula to represent $g(x) = \dfrac{4x-7}{2x^2+3x-2}$ as a power series centered at $c = 0$, and find the interval of convergence (domain) of this power series function.

More Ex. 4:

Still More Ex. 4:

Even More Ex. 4:

Ex. 5: Use the geometric series sum formula to represent $f(x) = \ln(1 - x^2)$ as a power series centered at $c = 0$, and find the interval of convergence (domain) of this power series function.

More Ex. 5:

Still More Ex. 5:

Ex. 6: Use the geometric series sum formula to represent $f(x) = \arctan(x)$ as a power series centered at $c = 0$, and find the interval of convergence (domain) of this power series function.

More Ex. 6:

Still More Ex. 6: